Machine Learning Approach to Mean Reversion Trading

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Mean-Reverting Prices

- Many asset prices exhibit mean reversion, such as stocks, commodities, volatility index, ETFs, fixed-income funds, etc.

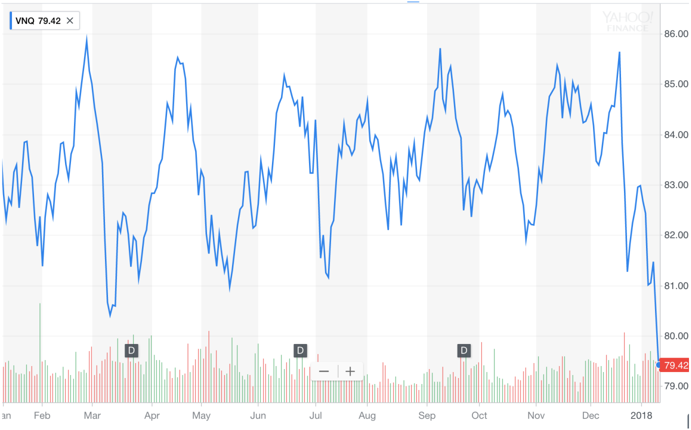

Figure: Vanguard REIT ETF (VNQ), Jan 2017 – Jan 2018.

Mean-Reverting Prices

- Mean reversion in a single asset tends to break down over the long term.

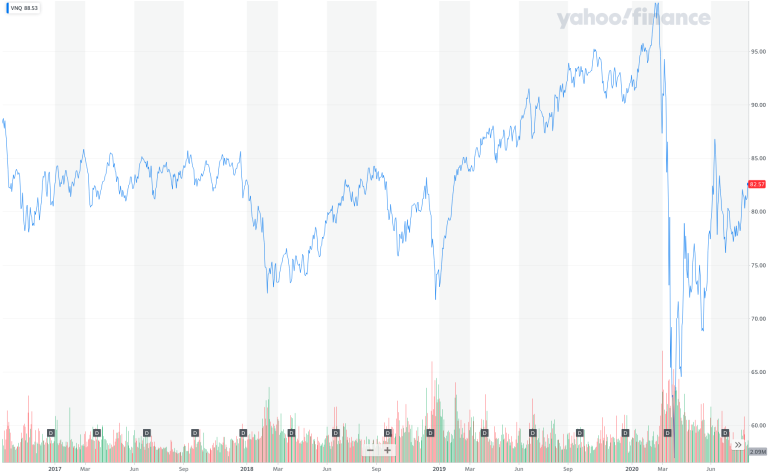

Figure: Vanguard REIT ETF (VNQ), Jan 2017 – Aug 10 2020.

Pairs Trading

- Pairs Trading has been widely used in industry for decades.
- Fund managers often attempt to construct mean-reverting prices via pairs trading.
- Traditional approach involves
  - finding stocks from the same industry/sector
  - computing correlations among assets
  - understanding the economic relationship between assets
- Examples:
  - GC, C, JPM, BAC, ... (banks)
  - ORCL, CSCO, ... (tech)
  - GC, SI (gold & silver futures)
  - BTC, ETH, LTC, BCH, ... (cryptos)

ETFs Pairs

- The proliferation of exchange-traded funds (ETFs) has facilitated pairs trading.
- ETFs, which traded on exchanges like stocks, are typically designed to track identical or similar indices/assets.
- Examples:
  - SPY, IVV, VOO (track S&P 500)
  - XLF, FNCL, VFH (financial sector)
  - FXI, VWO, EEM (emerging markets)
  - GLD, GDX, GDXJ (gold spot & miners)
  - GLD, SLV (gold & silver)

Pairs Trading Example

Mean-Reverting Model

- **Ornstein-Uhlenbeck (OU)**

$$dX_t = \mu(\theta - X_t)\,dt + \sigma\,dB_t,$$

where $\theta \in \mathbb{R}$ is the long-run mean, $\mu > 0$ is the speed of mean-reversion, and $\sigma > 0$ is the volatility parameter.

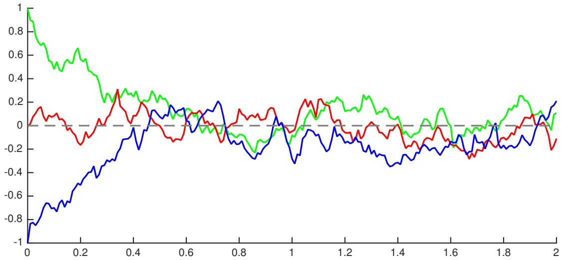

Figure: Simulated sample paths of OU processes with different initial values: (green) $X_0 = 1$, (red) $X_0 = 0$, (blue) $X_0 = -1$. Parameters: $\theta = 0$, $\mu = 5$, $\sigma = 0.5$.

Mean-Reverting Models

- **Ornstein-Uhlenbeck (OU)**

$$dX_t = \mu(\theta - X_t)\,dt + \sigma\,dB_t,$$

with parameters $\mu, \sigma > 0$, $\theta \in \mathbb{R}$.

- **Exponential OU (XOU)**

$$\xi_t = e^{X_t},$$

where X is the OU process.

- **Cox-Ingorsoll-Ross (CIR)**

$$dY_t = \mu(\theta - Y_t)\,dt + \sigma\sqrt{Y_t}\,dB_t, \quad Y_0 = y \geq 0,$$

with constants $\mu, \theta, \sigma > 0$.

Pairs Trading Example

- Construct a static portfolio in 2 risky assets, with portfolio value

$$X_t^{\alpha,\beta} = \alpha S_t^{(1)} - \beta S_t^{(2)}.$$

- Under the OU model, the conditional probability density is

$$f^{OU}(x_i|x_{i-1};\theta,\mu,\sigma) = \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} \exp\left(-\frac{(x_i - x_{i-1}e^{-\mu\Delta t} - \theta(1-e^{-\mu\Delta t}))^2}{2\tilde{\sigma}^2}\right),$$

where $\tilde{\sigma}^2 = \sigma^2 \frac{1-e^{-2\mu\Delta t}}{2\mu}$.

- With observed values $(x_i^{\alpha,\beta})_{i=0,1,\ldots,n}$, maximize the average log-likelihood

$$\ell(\theta,\mu,\sigma|x_0^{\alpha,\beta}, x_1^{\alpha,\beta},\ldots,x_n^{\alpha,\beta}) = -\frac{1}{2}\ln(2\pi) - \ln(\tilde{\sigma}) - \frac{1}{2n\tilde{\sigma}^2}\sum_{i=1}^{n}[x_i^{\alpha,\beta} - x_{i-1}^{\alpha,\beta}e^{-\mu\Delta t} - \theta(1-e^{-\mu\Delta t})]^2.$$

- For any α, find the strategy (α, β^*) and para. $(\theta^*,\mu^*,\sigma^*)$ from

$$\max_{\beta,(\theta,\mu,\sigma)} \ell(\theta,\mu,\sigma|x_0^{\alpha,\beta}, x_1^{\alpha,\beta},\ldots,x_n^{\alpha,\beta}).$$

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- For any α, find the strategy (α, β^*) and para. $(\theta^*, \mu^*, \sigma^*)$ from

$$\max_{\beta,(\theta,\mu,\sigma)} \ \ell(\theta,\mu,\sigma|x_0^{\alpha,\beta},x_1^{\alpha,\beta},\dots,x_n^{\alpha,\beta}).$$

Optimal Pairs

- The optimal parameters maximize the average log-likelihood. To express them, we define

$$X_x = \sum_{i=1}^{n} x_{i-1}^{\alpha,\beta}, \qquad X_y = \sum_{i=1}^{n} x_i^{\alpha,\beta},$$

$$X_{xx} = \sum_{i=1}^{n} (x_{i-1}^{\alpha,\beta})^2, \qquad X_{yy} = \sum_{i=1}^{n} (x_i^{\alpha,\beta})^2 \qquad X_{xy} = \sum_{i=1}^{n} x_{i-1}^{\alpha,\beta} x_i^{\alpha,\beta}, .$$

- The optimal parameter estimates under the OU model are given explicitly by

$$\theta^* = \frac{X_y X_{xx} - X_x X_{xy}}{n(X_{xx} - X_{xy}) - (X_x^2 - X_x X_y)},$$

$$\mu^* = -\frac{1}{\Delta t} \ln \frac{X_{xy} - \theta^* X_x - \theta^* X_y + n(\theta^*)^2}{X_{xx} - 2\theta^* X_x + n(\theta^*)^2},$$

$$(\sigma^*)^2 = \frac{2\mu^*}{n(1 - e^{-2\mu^* \Delta t})} (X_{yy} - 2e^{-\mu^* \Delta t} X_{xy} + e^{-2\mu^* \Delta t} X_{xx} - 2\theta^* (1 - e^{-\mu^* \Delta t})(X_y - e^{-\mu^* \Delta t} X_x) + n(\theta^*)^2 (1 - e^{-\mu^* \Delta t})^2).$$

Optimal Portfolio

- In turn, we denote by $\hat{\ell}(\theta^*, \mu^*, \sigma^*)$ the maximized average log-likelihood.
- For any α, we choose the portfolio (α, β^*), where

$$\beta^* = \underset{\beta}{\operatorname{argmax}}\, \hat{\ell}(\theta^*, \mu^*, \sigma^* | x_0^{\alpha,\beta}, x_1^{\alpha,\beta}, \ldots, x_n^{\alpha,\beta}).$$

- For example, suppose we invest A dollar(s) in asset $S^{(1)}$, so we long $\alpha = A/S_0^{(1)}$ shares. At the same time, we short $\beta = B/S_0^{(2)}$ shares in $S^{(2)}$, for $B/A = 0.001, 0.002, \ldots, 1$.
- The sign of the initial portfolio value equals to the sign of the difference $A - B$. Without loss of generality, we set $A = 1$.

Optimal Pairs

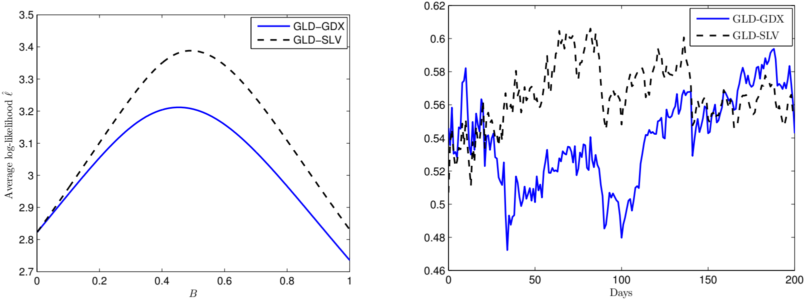


Figure: (left) Average log-likelihood $\hat{\ell}$ vs cash amount in asset $S^{(2)}$ $B := \beta S_0^{(2)}$. (right) Historical portfolio price paths: (i) $+\$1$ GLD and $-\$0.454$ GDX, (ii) $+\$1$ GLD and $-\$0.493$ SLV. Note: $\alpha = 1/S_0^{(1)}$.

Estimated Parameters

	Price	$\hat{\theta}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\ell}$
GLD-GDX	empirical	0.5388	16.6677	0.1599	3.2117
	simulated	0.5425	14.3893	0.1727	3.1304
GLD-SLV	empirical	0.5680	33.4593	0.1384	3.3882
	simulated	0.5629	28.8548	0.1370	3.3898

Table: MLE estimates of OU process parameters using historical prices of GLD, GDX, and SLV from August 2011 to May 2012. The portfolio consists of $\$1$ in GLD and -$\$0.454$ in GDX (resp. -$\$0.493$ in SLV). For each pair, the second row (simulated) shows the MLE parameter estimates based on a simulated price path corresponding to the estimated parameters from the first row (empirical).

Python package: `https://mlfinlab.readthedocs.io/en/latest/optimal_mean_reversion/ou_model.html`

Machine Learning Approach

Machine Learning Algo for MR Portfolios

- **Scalability**: Process more asset prices and identify more price patterns to discover trading opportunities
- **Speed**: Faster computation of optimal strategies, and faster execution
- **Adaptability**: Actively monitor performance and risk of strategies, adapting to live market conditions and price movements, to control exposure.
- **Potential**: more profits and lower cost

Problem Statement

Observations from previous approach:

- Pairs are pre-selected
- Limited to two assets, and must trade both assets (pairs trading)
- Parameters estimation is uncontrained

New objective: Given a set of m assets and their historical prices $S \in \mathbb{R}^{(T+1)\times m}$, design a portfolio that

- automatically selects from $m \geq 2$ assets (larger collection)
- but also prefers portfolios with fewer assets (parsimonious/sparse)
- is well-represented by an Ornstein-Uhlenbeck process (via MLE)
- but also controls the speed of mean reversion

OU-MLE

An OU process is defined by the SDE

$$dx_t = \mu(\theta - x_t)dt + \sigma dB_t.$$

where

- μ can be interpreted as speed of mean reversion,
- σ as level of volatility,
- θ as mean of x_t.

The likelihood of an OU process observed over a sequence $\{x_t\}_{t=1}^{T}$ is given by

$$\prod_{t=1}^{T} \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} \exp\left(-\frac{x_t - x_{t-1}\exp(-\Delta t\mu) - \theta(1-\exp(-\Delta t\mu))^2}{2\tilde{\sigma}^2}\right)$$

where $\tilde{\sigma}^2 = \sigma^2 \frac{1-\exp(-\Delta t\mu)}{2\mu}$.

Optimization (Minimization) Problem

- Let S be a $T+1 \times m$ matrix of asset prices, and $w \in \mathbb{R}^m$ be a vector of portfolio weights.
- Portfolio value times series: $x = Sw$.
- Minimizing the negative log likelihood with $x = Sw$ yields

$$\min_{\mu,\sigma^2,\theta,w} \frac{1}{2}\ln(2\pi) + \frac{1}{2}\ln(\tilde{\sigma}^2(\mu,\sigma^2)) + \frac{\|A(\mu)w - \theta(1-\exp(-\Delta t\mu))\mathbf{1}\|^2}{2T\tilde{\sigma}^2(\mu,\sigma^2)},$$

  with $A = S_{1:T} - \exp(-\Delta t\mu)S_{0:T-1}$, where the subscripts denote ranges for t.
- We can also limit the no. of assets, and find the best basket size.

Portfolio Weights Constraint

- Recall the function

$$\frac{1}{2}\ln(\tilde{\sigma}^2(\mu,\sigma^2)) + \frac{\|A(\mu)w - \theta(1-\exp(-\Delta t\mu))\mathbf{1}\|^2}{2T\tilde{\sigma}^2(\mu,\sigma^2)}$$

  where $\tilde{\sigma}^2 = \sigma^2 \frac{1-\exp(-\Delta t\mu)^2}{2\mu}$.
- If we set $\theta = 0, w = \vec{0}$, the second term disappears. Then the function value becomes **unbounded** if we take $\sigma^2 \to 0$.
- Hence we add a **constraint** $\|w\|_1 = 1$ to avoid unboundedness.

Re-writing the Problem

With a change of variables let

$$a = \tilde{\sigma}^2 = \frac{\sigma^2(1 - \exp(-2\Delta t\mu))}{2\mu},$$

$$c = \exp(-\Delta t\mu),$$

then the problem can be written more compactly as

$$\min_{a,c,\theta,\|w\|_1=1} \frac{1}{2}\ln(a) + \frac{1}{2Ta}\|A(c)w - \theta(1-c)\|^2.$$

Penalized MLE with Constraints

We further included two terms in the objective function

$$\min_{a,c,\theta,\|w\|_1=1,\|w\|_0\leq\eta} \frac{1}{2}\ln(a) + \frac{1}{2Ta}\|A(c)w - \theta(1-c)\|^2 + \gamma c$$

- $\|w\|_0 \leq \eta$ controls the total number of assets chosen for the portfolio. The total no. of assets used is capped at η (e.g. $\eta = 2, 4$, or 6).
- γc affects the mean-reverting speed of fitted OU-process via $c = \exp(-\Delta t \mu)$.

Algorithm

Our strategy is to first use variable projection as follows:

$$\begin{aligned} f(w,a,c,\theta) &= \frac{\ln(a)}{2} + \gamma c + \frac{\|A(c)w - \theta(1-c)\|^2}{2Ta} \\ f_1(w,a,c) &= \min_{\theta} f(w,a,c,\theta) \\ f_2(w,a) &= \min_{c} f_1(w,a,c) = \min_{c,\theta} f(w,a,c,\theta) \\ f_3(w) &= \min_{a} f_2(w,a) = \min_{a,c,\theta} f(w,a,c,\theta). \end{aligned}$$

and then use projected gradient descent to solve

$$\min_{\|w\|_1=1, \|w\|_0 \le \eta} f_3(w).$$

Remark: $f_3(w)$ is nonconvex and the constraints are also nonconvex.

Algorithm

- If we wish to use gradient descent to iterate, but it's possible that w_{i+1} is outside the feasible region $\mathcal{W}$. In the projected gradient descent, we simply choose the point nearest to w_{i+1} in the set $\mathcal{W}$.
- Once we obtain $f_3(w)$, we would like to know the projection onto $\mathcal{W} := \{z : \|z\|_1 = 1, \|z\|_0 \leq \eta\}$ to implement projected gradient descent.

$$\text{Proj}_{\mathcal{W}}(w) \leftarrow \text{argmin}_{\|z\|_1=1, \|z\|_0 \leq \eta} \|w - z\|^2$$

Algorithm 1 Projected Gradient Descent for $f_3(w; \gamma, \eta)$

Input: $w \in \mathbb{R}^m, S, f_3, \gamma, \eta$

$\mathcal{W} = \{w : \|w\|_1 = 1, \|w\|_0 \leq \eta\}$
for $i = 1, 2, 3, \ldots$ **do**
    $w \leftarrow \text{Proj}_{\mathcal{W}}(w - \delta_i \nabla_w f_3(w; \gamma, \eta))$
    $\text{loss}_i \leftarrow f_3(w; \gamma, \eta)$

Iterate till convergence.
(δ_i denotes stepsize chosen via line search.)

Algorithm

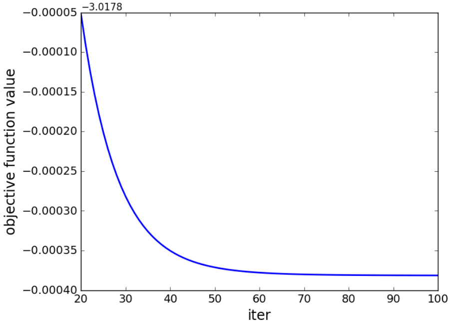


Figure: Objective function value's rate of decrease for our Algorithm. Rate of convergence of gradient descent is sublinear at $O(1/k)$, with $k =$ iterations.

Numerical results on real data

We performed experiments using closing stock prices during 2013-2018 from 3 groups of assets:

- Precious Metals: GLD, GDX, GDXJ, SLV, GG, ABX
- Large Capital Equities: GOOG, JNJ, NKE, MCD, SBUX, SPY, VIG, VO
- Oil Companies: BP, COP, CVX, OIL, USO, VLO, XOM

Numerical results on real data

Precious Metals (6 assets): GLD, GDX, GDXJ, SLV, GG, ABX

η	Selected Assets
2	GLD -0.17, SLV 0.83
4	GLD -0.08, GDX -0.21, SLV 0.44, ABX 0.27
6	GLD -0.07, GDX -0.29, GDXJ 0.03 SLV 0.30 GG 0.10 ABX 0.21

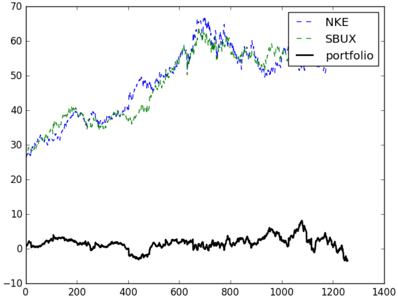
Large Cap (8 assets): GOOG, JNJ, NKE, MCD, SBUX, SPY, VIG, VO

η	Selected Assets
2	NKE -0.49, SBUX 0.51
4	JNJ -0.12 NKE -0.36, MCD -0.11, SBUX 0.41
6	JNJ -0.10, NKE -0.27, MCD -0.07, SBUX 0.36, SPY 0.12, VO -0.08

Energy (7 assets): BP, COP, CVX, OIL, USO, VLO, XOM

η	Selected Assets
2	OIL -0.59, USO 0.41
4	COP -0.01, CVX -0.02, OIL -0.57 USO 0.41
6	BP -0.01 COP -0.01, CVX -0.01, OIL -0.57, USO 0.41, VLO 0.002

Numerical results on real data: pair in Large capital equities



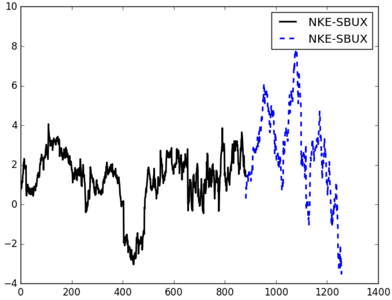


Table: Left: pair of assets and portfolio; right: zoomed-in portfolio

Numerical results on real data

Oil Companies: BP, COP, CVX, OIL, USO, VLO, XOM

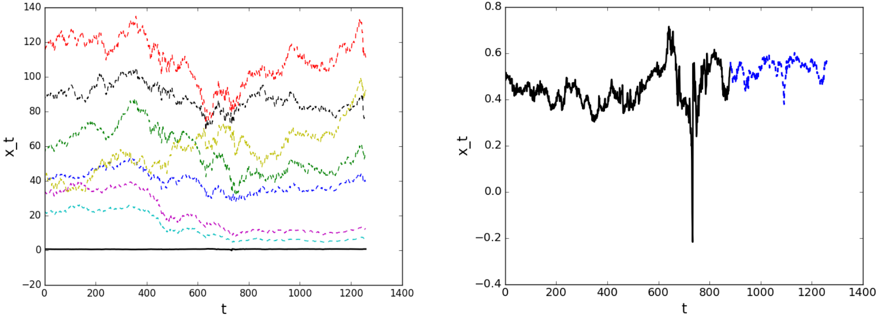


Table: Left: all assets and portfolio; right: zoomed-in portfolio

Portfolio Weights and Log-Likelihoods

Assets	2	4	6	indiv. nll (train,test)
GLD	-0.17	-0.08	-0.07	0.77,0.44
GDX		-0.21	-0.29	0.05,-0.30
GDXJ			0.03	0.70, 0.38
SLV	0.83	0.44	0.30	-0.69, -1.0
GG			0.10	-0.04, -0.44
ABX		0.27	0.21	-0.24 ,-0.54
Port.	-1.48,-1.72	-1.95,-2.12	-2.18,-2.35	
GOOG				2.66, 3.06
JNJ		-0.12	-0.10	0.40, 0.86
NKE	-0.49	-0.36	-0.27	0.09, 0.43
MCD		-0.11	-0.07	0.49, 1.09
SBUX	0.51	0.41	0.36	0.02, 0.05
SPY			0.12	0.95 ,1.00
VIG				-0.07 ,0.01
VO			-0.08	0.53 ,0.45
Port.	-0.70,-0.08	-0.77,-0.14	-1.07,-0.52	

Figure: Negative log-likelihood (nll) of assets groups for $\eta \in \{2, 4, 6\}$ and $\gamma = 0$. The bottom row shows the (training, testing) nll of our optimal portfolios.

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Thank you!

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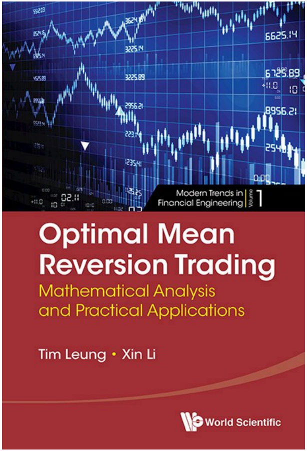

Figure: Optimal Mean Reversion Trading Mathematical Analysis and Practical Applications, 2016

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